ME 305 Fluid Mechanics I

Part 4

Integral Formulation of Fluid Flow

These presentations are prepared by

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Reynolds Transport Theorem

$$\frac{dN_{sys}}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho\,\eta\, d\forall + \int_{CS} \rho\,\eta\,(\vec V\cdot\vec n)dA$$

Rate of change of property N within the system. = Rate of change of property N in the corresponding CV (zero for steady flows) + Net rate at which property N exits the CV

Conservation Eqn	N	η	Left hand side
Mass	m	1	Zero
Linear Momentum	$\vec P$	$\vec V$	$\sum \vec F$
Angular Momentum	$\vec H$	$\vec r\times\vec V$	$\sum \vec T$
Energy	E	e	$\dot Q + \dot W$

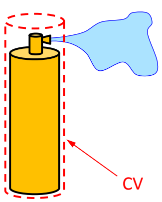
Conservation of Mass (Continuity Equation)

- Taking $N = m$ and $\eta = 1$, RTT becomes

$$\frac{dm_{sys}}{dt} = \boxed{0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho\,(\vec V\cdot\vec n)dA}$$

Rate of change of mass in the CV

Net rate of mass exiting through the CS



$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall < 0$: Mass inside the CV decreases in time

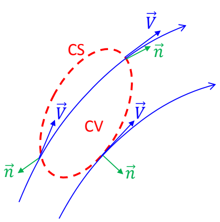
$\int_{CS}\rho\,(\vec V\cdot\vec n)dA > 0$: The amount of mass leaving the CV is in balance with the above term

Mass Flow Rate ($\dot m$)

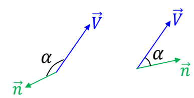
- The surface integral of the continuity equation is known as the mass flow rate ($\dot m$). It has the units of kg/s.

$$\boxed{\dot m = \int_{CS} \rho\,(\vec V\cdot\vec n)dA \quad \text{[kg/s]}}$$

Usually the CS is separated into parts and the integral over each part is evaluated one by one.



- $\dot m = \begin{cases} > 0 & \text{if } \vec V\cdot\vec n > 0 \quad \text{outflow} \\ = 0 & \text{if } \vec V\cdot\vec n = 0 \quad \vec V = 0 \text{ or flow is parallel to the CS} \\ < 0 & \text{if } \vec V\cdot\vec n < 0 \quad \text{inflow} \end{cases}$



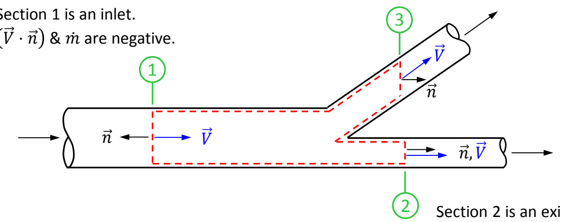
$\vec V\cdot\vec n = |\vec V|\cos(\alpha)$

Mass Flow Rate ($\dot{m}$)

- Consider the following pipe branching into two.

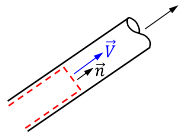
Section 3 is an exit.
$(\vec{V} \cdot \vec{n})$ & $\dot{m}$ are positive.

Section 1 is an inlet.
$(\vec{V} \cdot \vec{n})$ & $\dot{m}$ are negative.



Section 2 is an exit.
$(\vec{V} \cdot \vec{n})$ & $\dot{m}$ are positive.

Alternative CS
selection at section 3
(prefer this one)



Volumetric Flow Rate (Q) and Average Velocity ($\overline{V}$)

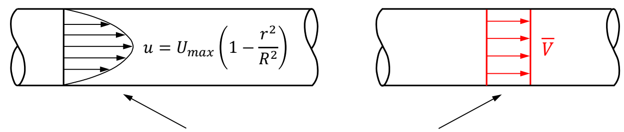
- If the density is uniform at a cross section of area A, volumetric flow rate (Q) is defined as

$$\boxed{Q = \frac{\dot{m}}{\rho} = \int_A (\vec{V} \cdot \vec{n}) dA \qquad [\mathrm{m^3/s}]}$$

- Average velocity over a section of a flow field is defined as

$$\overline{V} = \frac{\dot{m}}{\rho A} = \frac{Q}{A} \qquad (\rho \text{ is uniform over the section})$$

$$\boxed{\begin{aligned} \dot{m} &= \rho A \overline{V} \\ Q &= A \overline{V} \end{aligned}}$$



$\overline{V}$ is defined such that these two cases provide the same flow rate.

Volumetric Flow Rate and Average Velocity (cont'd)

Exercise: For the pipe example given in the previous slide find the relation between the centerline velocity and the average velocity.

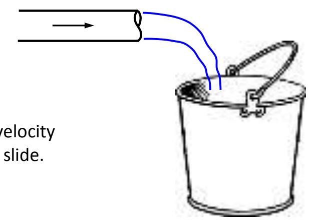
Exercise: Consider the water exiting a pipe of diameter $D = 2$ cm. To find the flow rate we measured the time it will take to fill a bucket of volume $V = 10$ liters. It took 65 s to fill it. Determine

a) the volumetric flow rate of water.

b) the mass flow rate of water.

c) average velocity at the exit.

d) the centerline velocity at the exit if the velocity profile is parabolic as given in the previous slide.



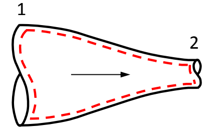
Continuity Equation for Steady Flows

- For a steady flow (time independent flow) partial time derivative of RTT is zero, and the continuity equation reduces to

$$\int_{CS} \rho \, (\vec{V} \cdot \vec{n}) dA = 0$$

- Continuity equation for a steady flow is $\displaystyle\sum_{inlets} \dot{m}_i + \sum_{outlets} \dot{m}_o = 0$

- For a common case of unidirectional, uniform flow with a single inlet and single exit



Continuity : $\underbrace{-\rho_1 A_1 V_1}_{\text{inlet}} + \rho_2 A_2 V_2 = 0$

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

Continuity Equation for Incompressible Flow

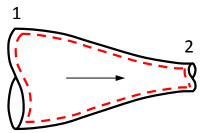
- For an incompressible flow (not necessarily steady) density is constant and continuity equation simplifies as

$$\rho \frac{\partial}{\partial t}\int_{CV} d\forall \; + \; \rho \int_{CS} (\vec{V}\cdot\vec{n})dA = 0$$

Zero since volume of a CV is constant

$$\rho \cancel{\frac{\partial(\forall_{CV})}{\partial t}} \; + \; \rho \int_{CS} (\vec{V}\cdot\vec{n})dA \;=\; 0 \qquad \rightarrow \qquad \int_{CS} (\vec{V}\cdot\vec{n})dA = 0$$

- For a common case of unidirectional, uniform flow with a single inlet and single exit



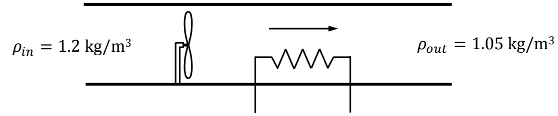
Continuity : $-A_1V_1 + A_2V_2 = 0$

$$A_1V_1 = A_2V_2$$

(true even if the flow is unsteady)

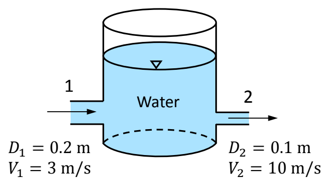
Mass Conservation Exercises (cont'd)

Exercise : A hair dryer can be considered as a duct of constant diameter with electric resistors in it. A small fan pulls the air in and forces it through the resistors where it is heated. If the density of air is 1.2 kg/m^3 at the inlet and 1.05 kg/m^3 at the outlet, determine the percent increase in the velocity of air as it flows through the dryer (Reference: Çengel's book).

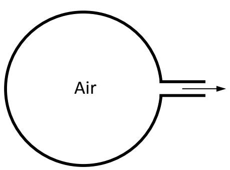


Mass Conservation Exercises (cont'd)

Exercise : A cylindrical tank, open at the top, has a diameter of 1 m. Water is flowing into the tank through pipe 1, and flowing out through pipe 2. Determine the rate at which the water level is rising / falling in the tank.

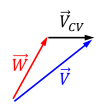


Exercise : A tank of 0.05 m^3 volume contains air at 800 kPa and 15 ºC. At $t = 0$, air begins escaping from the tank through a valve with a flow area of 65 mm². The air passing through the valve has a speed of 300 m/s and a density of 6 kg/m³. Determine the instantaneous rate of change of density in the tank at $t = 0$.



Moving CV

- Although we defined a CV to be fixed in space, some problems are solved easier by using a moving CV.
- In such a case, fluid velocity relative to the moving CV becomes important.
- Relative velocity ($\vec{W}$) is the fluid velocity seen by an observer moving with the CV. Absolute velocity ($\vec{V}$) is the fluid velocity seen by a fixed observer.



$\vec{V}_{CV}$: Velocity of the moving CV

$\vec{W}$: Relative velocity

$\vec{V}$: Absolute velocity

$$\vec{V} = \vec{W} + \vec{V}_{CV}$$

- Continuity equation takes the following form

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \, (\vec{W}\cdot\vec{n})dA$$

Moving CV (cont'd)

Exercise : An airplane moves in still air at a speed of 971 km/h. The frontal area of the jet engine is 0.8 m² and the entering air density is 1.2 kg/m³. The exhaust area is 0.558 m² and the exhaust gas density is 0.515 kg/m³. A stationary observer on the ground observes that the jet engine exhaust gases move with a speed of 1050 km/h. Estimate the mass flow rate of fuel into the engine (Reference: Munson's book).

Exercise : Water enters a rotating lawn sprinkler through its base at a steady rate of 1000 ml/s. The exit area of each of the two nozzles is 30 mm². Determine the average speed of the water leaving the nozzle, relative to the nozzle, if the rotary sprinkler head

a) is kept stationary,

b) rotates at 600 rpm,

c) accelerates from 0 to 600 rpm.

(Reference: Munson's book)

4-13

Deforming CV

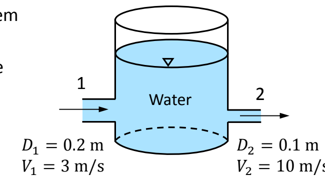
- Although we defined a CV to have fixed shape, some problems may also be solved by using a deforming CV with a moving CS.
- When the fluid passes through a moving part of the CS, relative velocity with respect to the moving CS should be used.
- Continuity equation becomes

Velocity relative to the CS

$$\vec{V} = \vec{W} + \vec{V}_{CS}$$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \, (\vec{W} \cdot \vec{n}) dA$$

Exercise: Solve the cylindrical tank problem of Slide 4-12 using a deforming CV such that the CS is attached to the free surface and goes up/down (deforms) with it.



4-14

Conservation of Linear Momentum

- Taking $N = \vec{P}$ and $\eta = \vec{V}$, RTT becomes

$$\frac{d\vec{P}_{sys}}{dt} = \boxed{\sum \vec{F} = \frac{\partial}{\partial t}\int_{CV} \rho \vec{V} d\forall + \int_{CS} \rho \vec{V}(\vec{V} \cdot \vec{n}) dA}$$

Sum of external forces acting on the CV

Rate of change of linear momentum in the CV

Net rate of flow of linear momentum through the CS

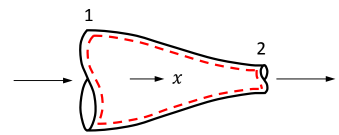
- Unlike the mass conservation equation, this one is a vector equation.
- For steady flows partial time derivative is zero. First term drops

$$\sum \vec{F} = \int_{CS} \rho \vec{V}(\vec{V} \cdot \vec{n}) dA$$

4-15

Conservation of Linear Momentum (cont'd)

- For the common case of steady, unidirectional, uniform flow with a single inlet and single exit



Continuity: $|\dot{m}| = \rho_1 A_1 V_1 = \rho_2 A_2 V_2$

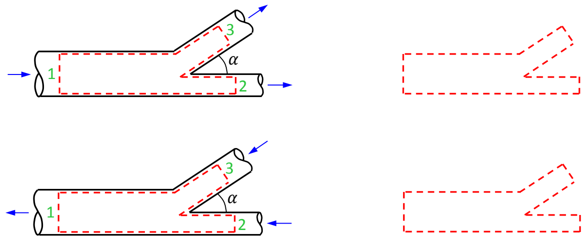
x component of linear mom.: $\sum F_x = \rho_2 V_2^2 A_2 - \rho_1 V_1^2 A_1 = |\dot{m}|(V_2 - V_1)$

4-16

Cons. of Linear Momentum (cont'd)

- Direction of momentum flow rate at the inlets/exits is always in the direction of the outward normal $\vec{n}$.
- In solving problems you can show them with arrows just like forces. I prefer double arrows (⇒) for them.

Exercise: Considering uniform flow at the inlets and the exits show the momentum flow rates and the forces on a separately drawn CV (like drawing a free body diagram).



Cons. of Linear Momentum (cont'd)

- Conservation of linear momentum is a vector equation.
- It can be written in component form. Cartesian components are

x component : $\sum F_x = \frac{\partial}{\partial t}\int_{CV} \rho u \, d\forall + \int_{CS} \rho u (\vec{V} \cdot \vec{n}) dA$

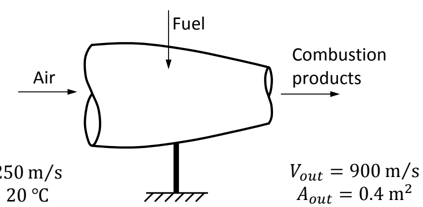
y component : $\sum F_y = \frac{\partial}{\partial t}\int_{CV} \rho v \, d\forall + \int_{CS} \rho v (\vec{V} \cdot \vec{n}) dA$

z component : $\sum F_z = \frac{\partial}{\partial t}\int_{CV} \rho w \, d\forall + \int_{CS} \rho w (\vec{V} \cdot \vec{n}) dA$

$\vec{V}$ remains as it is in the dot product

Linear Momentum Exercises

Exercise : A jet engine operates on a test stand with the shown inlet and exit conditions. Fuel-to-air mass ratio is 1/30. Calculate the horizontal force to be applied by the stand to hold the engine in its place. Assume that the fuel is injected in a vertical direction.

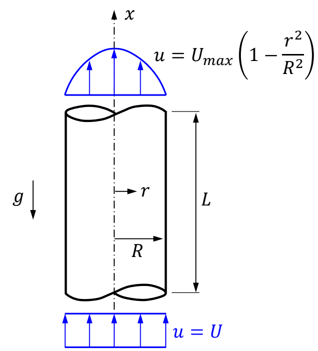


$V_{in} = 250$ m/s
$T_{in} = 20$ °C
$A_{in} = 0.5$ m^2
$P_{in} = 1$ atm

$V_{out} = 900$ m/s
$A_{out} = 0.4$ m^2
$P_{out} = 1$ atm

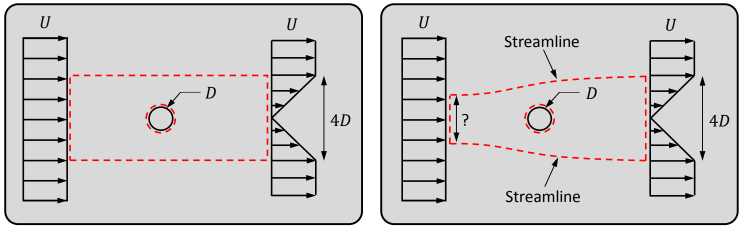
Linear Momentum Exercises (cont'd)

Exercise : Water flow inside a constant diameter pipe develops from uniform profile to parabolic profile. Contact force between the water and the pipe wall is F_z. Determine an expression for the pressure drop between the inlet and the exit.



Linear Momentum Exercises (cont'd)

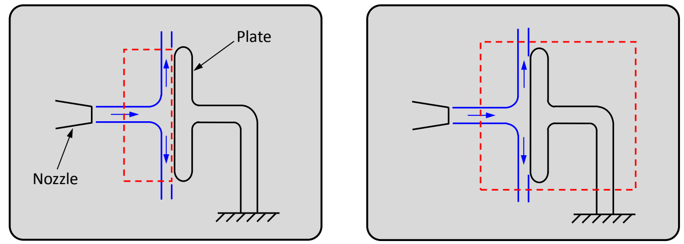
Exercise : A circular cylinder is immersed in a steady, incompressible flow. Velocities at the upstream and downstream locations are measured and simplified as follows. Calculate the drag force acting on the cylinder using the two control volumes shown below. Assume constant pressure everywhere (which is a questionable assumption) (Reference: Aksel's book).



4-21

Linear Momentum Exercises (cont'd)

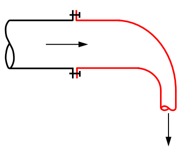
Exercise : Water exiting a stationary nozzle strikes a flat plate as shown. Water leaves the nozzle as a circular jet at 15 m/s. Nozzle exit area is 0.01 m^2. After striking the plate water leaves the plate tangentially as a circular disk. Determine the horizontal component of the anchoring force at the support. Solve the problem using two different CVs as shown. Which one is easier to use?



4-22

Linear Momentum Exercises (cont'd)

Exercise : Water flows steadily through the 90° reducing elbow. At the inlet to the elbow, the absolute pressure is 220 kPa and the cross sectional area is 0.01 m^2. At the outlet, cross sectional area is 0.0025 m^2 and the velocity is 16 m/s. The elbow discharges to the atmosphere. Determine the force required to hold the elbow in its place (Reference: Fox's book).



4-23

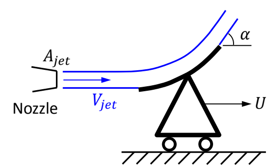
Linear Momentum – Moving CV

- Use relative velocity as follows

$$\vec{W} = \vec{V} - \vec{V}_{CV}$$

$$\sum \vec{F} = \frac{\partial}{\partial t}\int_{CV} \rho \vec{V} d\forall + \int_{CS} \rho \vec{V}(\vec{W}\cdot\vec{n})dA$$

Exercise : The vane with a turning angle of α moves at a constant speed of U. It receives a jet of water that leaves a stationary nozzle with speed V_{jet}. The nozzle exit area is A_{jet}. Area of the jet is assumed to remain constant along the vane. Determine the force components that act on the vane by the ground. Neglect the weights of the fluid and the vane.



$U = 10$ m/s
$V_{jet} = 30$ m/s
$A_{jet} = 0.003$ m^2
$\alpha = 60^o$

4-24

Conservation of Angular Momentum

- Taking $N = \vec{H}$ and $\eta = \vec{r} \times \vec{V}$, RTT becomes

$$\frac{d\vec{H}_{sys}}{dt} = \boxed{\sum \vec{T} = \frac{\partial}{\partial t}\int_{CV} \rho(\vec{r} \times \vec{V})d\forall \quad + \quad \int_{CS} \rho(\vec{r} \times \vec{V})(\vec{V} \cdot \vec{n})dA}$$

Sum of external torques acting on the CV

Rate of change of angular momentum in the CV

Rate of angular momentum exiting through the CS

- For a steady flow partial time derivative is zero. First term drops

$$\sum \vec{T} = \int_{CS} \rho(\vec{r} \times \vec{V})(\vec{V} \cdot \vec{n})dA$$

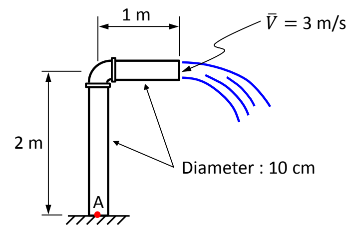
4-25

Conservation of Angular Momentum (cont'd)

Exercise : Underground water is pumped to the surface through a 10 cm diameter pipe that consists of a 2 m long vertical and 1 m long horizontal section. Water discharges to atmospheric air at an average velocity of 3 m/s. Mass of the horizontal pipe section when filled with water is 12 kg per meter length. The pipe is anchored on the ground by a concrete base. Determine

a) the moment acting at the base of the pipe (point A)

b) length of the horizontal section that would make the moment at point A zero.



4-26

Conservation of Angular Momentum – Moving CV

- Use relative velocity as follows $\vec{W} = \vec{V} - \vec{V}_{CV}$

$$\sum \vec{T} = \frac{\partial}{\partial t}\int_{CV} \rho(\vec{r} \times \vec{V})d\forall + \int_{CS} \rho(\vec{r} \times \vec{V})(\vec{W} \cdot \vec{n})dA$$

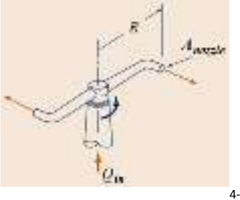
Exercise : Water enters a rotating lawn sprinkler through its base at the steady rate of 1000 ml/s. The exit area of each of the two nozzles is 30 mm² and the flow leaving each nozzle is in the tangential direction. The radius from the axis of rotation to the centerline of each nozzle is 200 mm.

a) Determine the resisting torque required to hold the sprinkler head stationary.

b) Determine the resisting torque associated with the sprinkler rotating with a constant speed of 500 rev/min.

c) Determine the speed of the sprinkler if no resisting torque is applied.

(Reference: Munson's book)



4-27

Conservation of Energy

- Taking $N = E$ and $\eta = e$, RTT becomes

$$\frac{dE_{sys}}{dt} = \boxed{\dot{Q} + \dot{W} \quad = \quad \frac{\partial}{\partial t}\int_{CV} \rho e d\forall \quad + \quad \int_{CS} \rho e(\vec{V} \cdot \vec{n})dA}$$

Rate of heat transfer into the system and rate of work done on the system

Rate of change of energy in the CV

Rate of energy exiting through the CS

- $\dot{Q} > 0$ if heat is added into the system.

  $\dot{W} > 0$ if work is done on the system.

- Specific energy, e, includes internal, kinetic and potential energy terms

$$e = \breve{u} + \frac{V^2}{2} + gz$$

4-28

Conservation of Energy (cont'd)

$$\dot{Q}+\dot{W} \;=\; \frac{\partial}{\partial t}\int_{CV}\rho\left(\breve{u}+\frac{V^2}{2}+gz\right)d\forall \;+\; \int_{CS}\rho\left(\breve{u}+\frac{V^2}{2}+gz\right)(\vec{V}\cdot\vec{n})dA$$

- Most common forms of work in fluid systems are
  - shear work : work done on the CS due to viscous stresses. Can be made zero by taking inlets and exits perpendicular to flow.
  - shaft work : e.g. energy delivered to the fluid by a pump or energy extracted from the fluid by a turbine.
  - flow work : energy necessary to push a certain amount of fluid into the CV or out of the CV. It depends on the pressure and velocity at an inlet or an exit.

$$\dot{W}_f=\int_A -p\;(\vec{V}\cdot\vec{n})dA$$

Conservation of Energy (cont'd)

- $\dot{W}_f$ is treated separately from other types of work and it is transferred to the right hand side.

Flow work

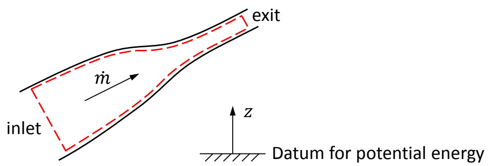
$$\dot{Q}+\dot{W} \;=\; \frac{\partial}{\partial t}\int_{CV}\rho\left(\breve{u}+\frac{V^2}{2}+gz\right)d\forall \;+\; \int_{CS}\rho\left(\underbrace{\breve{u}+\frac{p}{\rho}}_{h:\ \text{Enthalpy}}+\frac{V^2}{2}+gz\right)(\vec{V}\cdot\vec{n})dA$$

$\dot{W}$ no longer contains flow work

Flow work is accounted inside the enthalpy term

Conservation of Energy (cont'd)

- For a steady, single inlet - single exit flow with uniform properties at inlet and exit sections



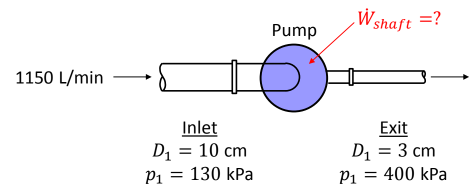
$$\dot{Q}+\dot{W}=|\dot{m}|\left[\left(h+\frac{V^2}{2}+gz\right)_{exit}-\left(h+\frac{V^2}{2}+gz\right)_{inlet}\right]$$

- Dividing both sides by the constant mass flow rate

$$q+w=\left(h+\frac{V^2}{2}+gz\right)_{exit}-\left(h+\frac{V^2}{2}+gz\right)_{inlet}$$

Conservation of Energy Exercises (cont'd)

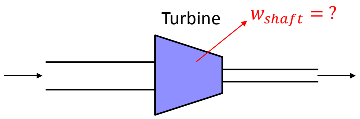
Exercise : A pump delivers water at a steady rate of 1150 L/min. Inlet and exit pipe diameters are 10 cm and 3 cm, respectively. Inlet and exit pressures are 130 kPa and 400 kPa, respectively. There is no significant elevation difference between the inlet and exit of the pump. The rise in internal energy of water, due to the temperature rise across the pump is 250 J/kg. Considering adiabatic pumping process determine the power delivered to the fluid by the pump.



$$\breve{u}_2-\breve{u}_1=250\text{ J/kg}$$

Conservation of Energy Exercises (cont'd)

Exercise : A steam turbine is used to produce electricity. Steam enters the turbine with a velocity of 30 m/s and enthalpy of 3500 kJ/kg. The steam leaves the turbine as a mixture of vapor and liquid having a velocity of 60 m/s and enthalpy of 2300 kJ/kg. Assuming adiabatic conditions and negligible elevation changes, determine the work output of the turbine per unit mass of fluid flowing.



Inlet
$V_1 = 30$ m/s
$h_1 = 3500$ kJ/kg

Exit
$V_2 = 60$ m/s
$h_1 = 2300$ kJ/kg